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ON THE COUPLING BETWEEN THREE-DIMENSIONAL BODIES AND SLIPPING CABLES

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Abstract—This paper examines the system obtained by coupling a three-dimensional body with a stretched cable in the case of frictionless contact, in order to evidence the consequences of some characteristic aspects such as the inseparable coupling arising between local and global deformation. The analysis is carried out for hyperelastic system components in the range of a finite deformation theory and furnishes a description of the kinematics, balance conditions and infinitesimal stability. The proposed formulation may be used to analyze numerous engineering techniques; the application of some results obtained to a real case is illustrated by a simple but meaningful example involving a beam. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION

The technique of introducing stretched cables slipping along a lubricated of frictionless path placed in the interior of a three-dimensional solid body or the other technique, which is similar under a qualitative point of view, of disposing stretched cables on the exterior of a body, constraining it to follow a path described by some points fixed on the body surface, holds increasing interest and permits the solving of numerous problems of different natures. Such a technique can, by way of example, be employed to control stress fields in solids, in order to reduce tensile stresses on fragile materials (masonry or concrete) or to oppose the stress field produced by external actions (beams or plates prestressed by external or internal slipping cables); in a similar manner, the technique permits the obtaining of a predetermined class of deformations or performing a displacement measurement of the body by controlling the cable state. Furthermore, the system may be adopted to eliminate the eventuality of disjunction in structures realized by linking in contact separate bodies (segmentally erected bridges and, more generally, unilateral constraint problems). In conclusion, the cable can be used as a sensor or an actuator for flexible structures and the system has interesting applications in structural mechanics, robotics and monitoring techniques.

The most characteristic aspect of the system described consists of the coupling between the local cable strain and the global body deformation. This is due to the slip of the cable and permits the provision of global effects on the body by controlling the cable strain at a point, e.g. at one end.

Despite this wide field of application, the works found in literature mainly regard structural mechanics [only by way of example, the work of Hoadley (1963) and the more recent works of Saadatmanesh *et al.* (1989) and Virlogeux (1990)] and are exclusively devoted to solving a number of specific problems, but they neither furnish a unitary treatment of the subject nor do they sufficiently evidence the characteristic aspects deriving from the inseparable coupling between local strain and global strain provided by the slipping cable.

This work intends to provide a rather general formulation of the static problem by starting from the kinematical analysis of the system obtained by coupling a three-dimensional body and a cable in the range of the non linear theory of continua. More precisely, it was assumed that the cable is constrained to follow a path traced by a continuous curve, or by some fixed points, rigidly linked to the body, and it is free to slip along this path. When slipping leads to homogeneous strain of the cable, it becomes possible to show that the positions of every material point of the cable after deformation can be deduced from the deformation of the three-dimensional body only and, consequently, the latter can be assumed as the sole kinematical descriptor of the whole coupled system.

The analysis continues by determining the balance conditions by requiring the critical condition for the total potential energy. In particular, it was assumed that the external actions possess a potential, the materials are hyperelastic and the homogeneous strain in the cable was obtained as a consequence of its homogeneity and the frictionless contact. The dependence of the cable strain on the global body strain makes the weak formulation the only convenient formulation (Dall'Asta and Dezi, 1993) and the expression obtained permits developing some interesting observations with regard to failure behaviour, the solution with rectilinear cables and the problem concerning the choice and the complete characterization of a reference configuration.

Finally, the relevant question of the analysis of the stability of balanced solutions is examined. Interest in these specific systems arises from the observation that, contrary to many other elastic systems, in this case there is not a natural configuration with no stresses in the absence of forces in the neighborhood of which the problem is well posed, but all the actual configurations show a stress field provided by the interaction between its components, even if external forces are absent; this makes the stability analysis always relevant, at least with regard to infinitesimal perturbations. The infinitesimal stability condition was determined for a generic balanced configuration with finite deformation and some qualitative conclusions regarding the role played by the stretched cable can be deduced from the expression obtained. Among other things, it was proved that a substantial difference exists, with regard to stability, between the case of configurations with rectilinear cable slipping on a path linked to the body and configurations with rectilinear cable anchored at the ends only, even if the two cases furnish the same balanced stress fields and both may satisfy the same design instances.

In those systems where the interaction between stretched cable and body involves a very small deformation, it is possible to establish a linear relation between cable traction force and body stress field. In this case the stability of the system is only related to its geometry and this makes it possible to search for those cable paths which prevent instability phenomena altogether, i.e. for every value of the traction force on the cable.

To conclude, an illustrative application is reported; this describes the consequences of the different position of a cable anchored at the ends of a rectilinear beam with torsional-bending behaviour in the two extreme cases of cable linked at the ends only and cable constrained to a path for its whole length.

2. KINEMATICS

This first paragraph provides a description of kinematics for a system generated by means of the coupling between a slipping cable and a three-dimensional body. More precisely, it is assumed that the cable ends are linked to two material points of the body and is constrained to trace a path which is defined by a curve rigidly joined to the body in an initial tract and defined by a rectilinear segment connecting the end of the curve to the second anchorage point in the remaining tract (Fig. 1). This path identifies only the spatial curve described by the cable although its material points can slip along the path. The formulation proposed can easily be extended to more complex situations where the cable is constrained to follow paths defined by a greater number of points and curves. The kinematical analysis is completed by examining the case of homogeneous cable strain. In this case, it is possible to show that the cable deformation can be deduced from the body deformation and consequently, the latter can be adopted to describe the kinematics of the whole system.

Let body $\mathscr{B}$ be a three-dimensional manifold whose particles are identified by means of the three material co-ordinates (X_i; $i = 1, 2, 3$) coinciding, for assumption, with the co-ordinates (x_i) with respect to the orthonormal basis $\{\mathbf{A}_i\}$ which localize their positions $\mathbf{P}(X_i) = x_i\mathbf{A}_i = X_i\mathbf{A}_i$ in the reference configuration, letting Ω be the domain $\mathbf{P}(\mathscr{B})$ occupied by the body.

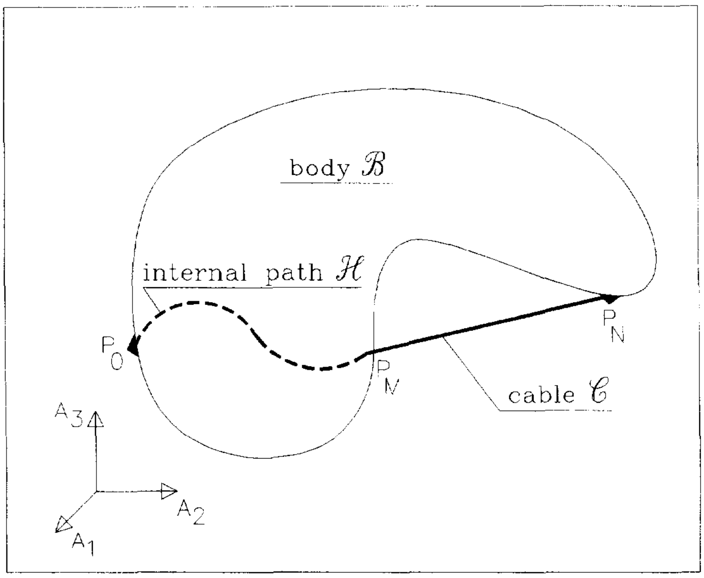

Fig. 1. Cable-body system.

Furthermore, let $\mathscr{H} = \{X_i = H_i(\eta)\,;\ \eta \in [0, m]\}$ be a subset of $\mathscr{B}$ which describes the regular curve $\mathbf{H}(\eta) = H_i(\eta)\mathbf{A}_i$ in the reference configuration. The tangent vector $\mathbf{H}_{,\eta}$ is defined everywhere, its modulus $|\mathbf{H}_{,\eta}|$ is always positive and the unit tangent vector is denoted by $\mathbf{G}(\eta) = \mathbf{H}_{,\eta}(\eta)/|\mathbf{H}_{,\eta}(\eta)|$. The indexes O, M and N are used for labeling the values assumed by a generic quantity at the two material points (X_{Oi}), (X_{Mi}), located at the ends of the curve, and at the third material point (X_{Ni}), located out of the curve. In the reference configuration these points occupy the positions $\mathbf{P}_O = \mathbf{H}(0) = X_{Oi}\mathbf{A}_i$, $\mathbf{P}_M = \mathbf{H}(m) = X_{Mi}\mathbf{A}_i$ and $\mathbf{P}_N = X_{Ni}\mathbf{A}_i$. The points (X_{Mi}) and (X_{Ni}) lie on the boundary of $\mathscr{B}$ and $\mathbf{G}$ denotes the unit vector oriented as the vector $\mathbf{P}_N - \mathbf{P}_M$, assumed external to the body.

Let the cable $\mathscr{C}$ be a uni-dimensional manifold whose material points are identified by means of the material co-ordinate $\rho \in [0, n]$ and the cable describes the following curve $\mathbf{R}(\rho) = \mathbf{R}_i(\rho)A_i$ in the reference configuration:

$$\mathbf{R}(\rho) = \mathbf{H}(\rho) \qquad \rho \in [0, m] \tag{1a}$$

$$\mathbf{R}(\rho) = \mathbf{P}_M + (\rho - m)\mathbf{G} \quad \rho \in (m, n] \tag{1b}$$

The function $\mathbf{E}(\rho)$ denotes the unit tangent vector $\mathbf{R}_{,\rho}/|\mathbf{R}_{,\rho}|$, coinciding with $\mathbf{G}$ when $0 \leqslant \rho \leqslant m$ and with $\bar{\mathbf{G}}$ when $m < \rho \leqslant n$. It follows that the ends of the cable occupy the same positions of the body particles (X_{Oi}) and (X_{Ni}) while those cable particles with $\rho \in [0, m]$ are located on the curve $\mathbf{H}(\eta)$. It may be useful to remark that the choice of assuming ρ and η such as to define the same position in the reference configuration descends from the observation that usually the more convenient way for expressing a parametrized space curve is unique; it should, however, not be forgotten that ρ and η are two substantially different quantities because the former denotes a material particle of the cable while the latter is a parameter denoting a material particle of the body, through the functions $H_i(\eta)$. From this it follows that the previous correspondence will be lost in configurations which are different from the one assumed as reference. The length of the cable from its former anchorage point $\mathbf{P}_O$ to the generic material particle ρ is described by means of the function

$$\Lambda(\rho) = \int_0^{\rho} |\mathbf{R}_{,\xi}|\,\mathrm{d}\xi \quad [0, n] \tag{2}$$

A generic configuration of the system can be described by means of the two deformation functions $\mathbf{p}(X_k) = p_i(X_k)\mathbf{A}_i$ and $\mathbf{r}(\rho) = r_i(\rho)\mathbf{A}_i$ establishing the position of the body $\mathscr{B}$ and the cable $\mathscr{C}$ by starting from the position in the reference configuration. The deformation description in the neighborhood of a body material point is furnished by the quantity $\nabla\mathbf{p}(X_k) = p_{i,j}(X_k)\mathbf{A}_i \otimes \mathbf{A}_j$ and it is here assumed that $\mathbf{p}$ be orientation preserving the locally invertible, at least almost everywhere, on Ω and along the cable path, i.e. $\det(\nabla\mathbf{p}) > 0$, $|\nabla\mathbf{p}\mathbf{H}_{,\eta}| > 0$, $|\mathbf{p}_N - \mathbf{p}_M| > 0$. The complex question of global invertibility is beyond the scope of this paper.

It is evident that the deformation $\mathbf{p}(\mathscr{H})$ of the curve $\mathscr{H}$ rigidly linked to the body can be deduced through the functions H_i and $\mathbf{h}(\eta) = \mathbf{p}(H_i(\eta))$ will denote the vector function describing this deformed curve. The derivative with respect to the parameter η can also be related to the body deformation

$$\mathbf{h}_{,\eta}(\eta) = \nabla\mathbf{p}(H_i(\eta))\mathbf{H}_{,\eta}(\eta) \tag{3}$$

while its modulus $|\mathbf{h}_{,\eta}|$ is positive for the previous assumption on the local invertibility.

With reference to cable deformation, it is assumed that $\mathbf{r}$ is orientation preserving and locally invertible, i.e. $|\mathbf{r}_{,\rho}| > 0$, at least almost everywhere and describes a path traced by the curve $\mathbf{h}$ in an initial tract and by the segment $\mathbf{p}_N - \mathbf{p}_M$ in the remaining tract, even if the correspondence between the parameter η and material co-ordinate ρ is not preserved. The local deformation of the cable is described by means of $\mathbf{e}(\rho) = \mathbf{r}_{,\rho}/|\mathbf{R}_{,\rho}|$ identifying the vector provided by the transformation of the tangent vector $\mathrm{E}(\rho)$ and is measured by means of the quantity $a(\rho) = |\mathbf{r}_{,\rho}|/|\mathbf{R}_{,\rho}|$ denoting the local stretch, so that $\mathbf{r}_{,\rho} = |\mathbf{R}_{,\rho}|\mathbf{e}$ or alternatively, by introducing the unit tangent vector in the deformed configuration $\hat{\mathbf{e}}(\rho) = \mathbf{r}_{,\rho}/|\mathbf{r}_{,\rho}|$, $\mathbf{r}_{,\rho} = |\mathbf{R}_{,\rho}|a\,\hat{\mathbf{e}}$.

Attention is now focused on those deformations satisfying the following particular condition:

$$a_{,\rho} = 0 \tag{4}$$

which describes a situation characterized by homogeneous strain of the cable. From this condition it follows that a must coincide with its mean value furnished by the ratio between the total length $l(\mathbf{p})$ of the path in the deformed configuration and the total length $L = \Lambda(n)$ of the path in the reference configuration, where both the quantities can be derived from the body configuration by means of the two relations:

$$L = \int_0^m |\mathbf{H}_{,\eta}|\,\mathrm{d}\eta + |\mathbf{P}_N - \mathbf{P}_M| \tag{5}$$

$$l(\mathbf{p}) = \int_0^m |\nabla\mathbf{p}(H_i(\eta))\mathbf{H}_{,\eta}(\eta)|\,\mathrm{d}\eta + |\mathbf{p}_N - \mathbf{p}_M| \tag{6}$$

so that

$$a = \frac{l(\mathbf{p})}{L} \tag{7}$$

Consequently, the strain measure a and $|\mathbf{r}_{,\rho}|$ are strictly positive as a consequence of the regularity assumed for p and there is no need to introduce $|\mathbf{r}_{,\rho}| > 0$ as an independent assumption.

Furthermore, it should be observed, as a peculiar characteristic of this system, that the local strain measure a of the cable is not related to the deformation of its neighborhood but depends on the global deformation of the body and cannot be expressed by an algebraic law but requires a functional dependence involving both the deformation $\mathbf{p}$ and the initial geometry described by H_i and (X_{Ni}).

In this case, it is also possible to obtain a complete description of the cable deformation **r** from the body deformation **p**. In particular, in order to provide the final position of each cable point ρ, the following function is introduced:

$$\chi(\eta) = \int_0^\eta |\nabla \mathbf{p}(\xi)\mathbf{H}_{,\eta}(\xi)|\,\mathrm{d}\xi \tag{8}$$

This measures the length of the path linked to the body from the material point $(H_i(0))$ to $(H_i(\eta))$, in the deformed configuration, its derivative satisfies the inequality $\chi_{,\eta}(\eta) > 0$ and the function is invertible. The homogeneity of the strain ensures the equality of the two ratios $\chi(\eta)/l$ and $\Lambda(\rho)/L$ for that cable material particle ρ lying at the point $\mathbf{p}(H_i(\eta))$ of the curve $\mathscr{H}$, so that its inverse χ^{-1} provides the following relation:

$$\eta = \chi^{-1}(a\Lambda(\rho)) \tag{9}$$

between the cable points and the body curve points. By introducing $\bar{\rho} = \Lambda^{-1}(\chi(m)/a)$ that identifies the cable material particle lying at the end $\mathbf{h}(m)$ of the body curve after the deformation, it becomes possible to reconstitute the position of each cable point by means of the relations

$$\mathbf{r}(\rho) = \mathbf{p}(H_i(\chi^{-1}(a\Lambda(\rho)))) \qquad \rho \in [0, \bar{\rho}] \tag{10a}$$

$$\mathbf{r}(\rho) = \mathbf{p}_M + \frac{\Lambda(\rho) - \Lambda(\bar{\rho})}{L - \Lambda(\bar{\rho})}(\mathbf{p}_N - \mathbf{p}_M) \quad \rho \in (\bar{\rho}, n]. \tag{10b}$$

so that the functional relation furnishing **r** from **p** has been made explicit and function **p** becomes the only kinematical descriptor of the whole cable-body system.

3. BALANCE CONDITIONS

In order to achieve a global balance condition for the system it is assumed that the body consists of a simple hyperelastic material (Tresdell and Noll, 1965) and, consequently, at every material point the positive real valued function $w(X_i, \mathbf{C})$ is defined and describes the density of elastic potential energy by starting from the local strain measure $\mathbf{C} = (\nabla \mathbf{p})^T \nabla \mathbf{p}$. This energy may tend to infinity when $|\nabla p|$ tends to zero or to infinity. Once w is known, it is possible to derive the stress measure furnished by the first and second Piola-Kirchhoff stress tensors, respectively denoted by $\mathbf{S}$ and $\bar{\mathbf{S}}$, by means of the following relations:

$$\mathbf{S}(\nabla \mathbf{p}) = \nabla \mathbf{p} \bar{\mathbf{S}}(\nabla \mathbf{p}) = \frac{\partial w}{\partial (\nabla \mathbf{p})} = \nabla \mathbf{p} \frac{\partial w}{\partial \mathbf{C}} \tag{11}$$

It is also assumed that the cable be hyperelastic and homogeneous, in the sense that a positive real valued function $\omega(\mathbf{e})$, which this time does not depend on the point ρ, is defined and furnishes the elastic potential energy for unit length, measured in the reference configuration, by starting from the local deformation measure, described by $\mathbf{e} = a\hat{\mathbf{e}}$. In particular, the request of frame indifference for ω leads to the conclusion that ω can depend on a only. As previously mentioned, it is accepted that ω may diverge to infinity when its argument a tends to zero or infinity. The derivative $t(a)$ of $\omega(a)$ with respect to a denotes the intensity of the internal force produced in the cable by the strain and the derivative with respect to **e** provides a vector $\mathbf{t}(\mathbf{e})$, tangent to the cable path, which completely describes this force, i.e.

$$\mathbf{t}(\mathbf{e}) = t(a)\hat{\mathbf{e}} = \omega_{,a}\hat{\mathbf{e}} \tag{12}$$

It is now assumed that the contact between body and cable occurs without friction or other tangential forces and no external forces act on the cable so that the stress state satisfies the condition

$$\mathbf{t}_{,\rho}(\rho)\cdot\mathbf{t}(\rho) = (\omega_{,aa}a_{,\rho}\hat{\mathbf{e}}+\omega_{,a}\hat{\mathbf{e}}_{,\rho})\cdot\omega_{,a}\hat{\mathbf{e}} = 0 \tag{13}$$

from which, taking into account that $\hat{\mathbf{e}}$ is a unit vector and assuming a monotone constitutive law ($\omega_{,aa} > 0$), it can be deduced that the previously analyzed condition $a_{,\rho} = 0$ must be verified. Therefore the homogeneous strain condition is a consequence of both the assumption of frictionless contact and cable homogeneity and this permits asserting that the potential elastic energy contained in the cable is constant along the cable, even if the internal force $\mathbf{t}$ may vary in direction. It follows that the total elastic energy of the deformed cable can be deduced simply by multiplying the initial length by the energy density evaluated for the strain a, depending on the global body deformation, and this calculation does not require the explicit knowledge of the local deformation expressed by $\mathbf{r}_{,\rho}$ or $\mathbf{e}$, even if this may however be obtained from $\mathbf{p}$ by means of χ^{-1}.

The external actions on the body consist of mass conservative forces $\mathbf{b}(X_i;\mathbf{p})$ equipped with a potential $\phi_b(X_i;\mathbf{p})$ and acting on Ω, contact conservative forces $\mathbf{f}(X_i;\nabla\mathbf{p})$ equipped with a potential $\phi_f(X_i,\mathbf{p},\nabla\mathbf{p})$ and acting on the surface described by a portion $\partial\Omega_s$ of the boundary of Ω (Ciarlet, 1988), while the points lying on the remaining boundary portion $\partial\Omega_u$ cannot move and maintain the position occupied in the reference configuration. In the static case, the total potential energy Ψ of the system exists and can be written in the following form, for a generic deformation $\mathbf{p}$ belonging to the space $U(\Omega)$ of the admissible deformations satisfying the boundary conditions on $\partial\Omega_u$:

$$\Psi(\mathbf{p}) = \int_\Omega [w(X_i;\nabla\mathbf{p}(X_i))-\mu_0(X_i)\phi_b(X_i,\mathbf{p})]\,\mathrm{d}\Omega+L\omega(a(\mathbf{p}))-\int_{\partial\Omega_s}\phi_f(X_i;\mathbf{p},\nabla\mathbf{p})\,\mathrm{d}\,\partial\Omega_s \tag{14}$$

where μ_0 denotes the mass density in the reference configuration. It is recalled that a is a functional of $\mathbf{p}$, and depends on the values assumed by the deformation and deformation gradient along its whole path (4–6) so that the simplicity of the previous expression is only formal.

At this point, it is possible to state the problem in its weak form by seeking the expression of the differential of Ψ, by assuming that the solutions coincide with the critical points of the functional Ψ. With regard to the term related to the cable it is observed that the variation of the energy density resulting from a deformation $\boldsymbol{\pi}\in U$ can be expressed by

$$\delta\omega(\mathbf{p};\boldsymbol{\pi}) = t(a(\mathbf{p}))\delta a(\mathbf{p};\boldsymbol{\pi}) \tag{15}$$

where the strain variation a can be evaluated considering its functional dependence on the trace of $\mathbf{p}$ of the curve $\mathbf{H}$ and has the following expressions:

$$\delta a(\mathbf{p};\boldsymbol{\pi}) = \left[\int_0^m \frac{\nabla\mathbf{p}(H_i(\eta))\mathbf{H}_{,\eta}(\eta)}{|\nabla\mathbf{p}(H_i(\eta))\mathbf{H}_{,\eta}(\eta)|}\cdot(\nabla\boldsymbol{\pi}(H_i(\eta))\mathbf{H}_{,\eta}(\eta))\,\mathrm{d}\eta+\frac{\mathbf{p}_N-\mathbf{p}_M}{|\mathbf{p}_N-\mathbf{p}_M|}\cdot(\boldsymbol{\pi}_N-\boldsymbol{\pi}_M)\right]\Big/L \tag{16}$$

In conclusion, the critical condition for the total potential energy has the form

$$\delta\Psi(\mathbf{p};\boldsymbol{\pi}) = \int_\Omega[\mathbf{S}\cdot\nabla\boldsymbol{\pi}-\mu_0\mathbf{b}\cdot\boldsymbol{\pi}]\,\mathrm{d}\Omega+t(a)\left[\int_0^m\frac{\nabla\mathbf{p}\mathbf{H}_{,\eta}}{|\nabla\mathbf{p}\mathbf{H}_{,\eta}|}\cdot\nabla\boldsymbol{\pi}\mathbf{H}_{,\eta}\,\mathrm{d}\eta+\frac{\mathbf{p}_N-\mathbf{p}_M}{|\mathbf{p}_L-\mathbf{p}_M|}\cdot(\boldsymbol{\pi}_N-\boldsymbol{\pi}_M)\right]$$
$$-\int_{\partial\Omega_s}\mathbf{f}\cdot\boldsymbol{\pi}\,\mathrm{d}\,\partial\Omega_s = 0 \quad \forall\boldsymbol{\pi}\in U \tag{17}$$

It is not within the aim of this paper to analyze questions regarding the existence of the solution, however, a minimum regularity requirement is assumed by introducing the following space for admissible deformations:

$$U(\Omega) = \Bigg\{\mathbf{p} : \mathbf{p} = 0 \text{ on } \partial\Omega_u ;$$

$$\|\mathbf{p}\|_U = \left[\int_\Omega |\nabla\mathbf{p}|^2 + |\mathbf{p}|^2 \, d\Omega + \int_0^m |\nabla\mathbf{p}\mathbf{H}_{,\eta}|^2 \, d\eta + |\mathbf{p}_N - \mathbf{p}_M|^2 \right]^{1/2} < \infty \Bigg\} \tag{18}$$

In this ambient one can expect to prove existence results under physically acceptable assumptions for w, ω, e.g. policonvexity, and for sufficiently regular actions **f** and **b** (Ciarlet, 1988). It is observed that posing the problem on too regular spaces might not furnish a good model because extreme deformations might be expected to occur at the interface between uni-dimensional cable and three-dimensional body.

Furthermore, it should be noted that it is not possible to derive a local equivalent formulation for this specific problem, as for classical solid theory by way of the Green formula, because the slipping cable introduces an inseparable coupling between local and global deformation. Even in the particular case in which body deformation can be pulled back to a function defined on a uni-dimensional domain as the cable deformation domain, e.g. Kirchhoff rod, the integration by part of the balance condition does not lead to a really local condition of differential type but furnishes an integro-differential condition which preserves the global structure (Dall'Asta and Dezi, 1993).

It is interesting and useful for the rest of the paper to provide an alternative formulation in which the system state is described by the displacement field $\mathbf{u} = \mathbf{p} - \mathbf{P}$ measured with respect to the reference configuration and the local pure strain is evaluated by means of the following objective quantities:

$$\mathbf{D}(\mathbf{u}) = ((\nabla\mathbf{p})^T\nabla\mathbf{p} - \mathbf{I})/2 = ((\nabla\mathbf{u})^T + \nabla\mathbf{u} + (\nabla\mathbf{u})^T\nabla\mathbf{u})/2 \tag{19}$$

$$d(\mathbf{u}) = a(\mathbf{p}) - 1 = \left[\int_0^m |(\mathbf{I} + \nabla\mathbf{u}(H_i(\eta)))\mathbf{H}_{,\eta}(\eta)| \, d\eta + |\mathbf{P}_N + \mathbf{u}_N - \mathbf{P}_M - \mathbf{u}_M| - L \right] \Big/ L \tag{20}$$

The variation of **D** caused by a variation of the displacement field has the form

$$\delta\mathbf{D}(\mathbf{u}; \mathbf{v}) = ((\nabla\mathbf{v})^T + \nabla\mathbf{v} + (\nabla\mathbf{v})^T\nabla\mathbf{u}) + (\nabla\mathbf{u})^T\nabla\mathbf{v})/2 \tag{21}$$

while, by introducing in advance the unit vectors tangent to the deformed cable path

$$\mathbf{g}(\mathbf{u}; \eta) = \frac{(\mathbf{I} + \nabla\mathbf{u}(H_i(\eta)))\mathbf{H}_{,\eta}(\eta)}{|(\mathbf{I} + \nabla\mathbf{u}(H_i(\eta)))H_{,\eta}(\eta)|} \tag{22}$$

$$\bar{\mathbf{g}}(\mathbf{u}) = \frac{\mathbf{P}_N - \mathbf{P}_M + \mathbf{u}_N - \mathbf{u}_M}{|\mathbf{P}_N - \mathbf{P}_M + \mathbf{u}_N - \mathbf{u}_M|} \tag{23}$$

the variation of a can be written in the form:

$$\delta a(\mathbf{u}; \mathbf{v}) = \left[\int_0^m \mathbf{g}(\mathbf{u}) \cdot \nabla\mathbf{v}\mathbf{H}_{,\eta} \, d\eta + \bar{\mathbf{g}}(\mathbf{u}) \cdot (\mathbf{v}_N - \mathbf{v}_M) \right] \Big/ L \tag{24}$$

and the critical condition for the total energy is expressed by the following relation:

$$\delta\Psi(\mathbf{u};\mathbf{v}) = \int_\Omega [\bar{\mathbf{S}}(\mathbf{D}(\mathbf{u}))\cdot((\nabla\mathbf{v})^T+\nabla\mathbf{v}+(\nabla\mathbf{v})^T\nabla\mathbf{u}+(\nabla\mathbf{u})^T\nabla\mathbf{v})/2-\mu_0\mathbf{b}(\mathbf{u})\cdot\mathbf{u}]\,\mathrm{d}\Omega$$

$$+t(d(\mathbf{u}))\left[\int_0^m \mathbf{g}(\mathbf{u})\cdot\nabla\mathbf{v}\mathbf{H}_{,\eta}\,\mathrm{d}\eta+\bar{\mathbf{g}}(\mathbf{u})\cdot(\mathbf{v}_N-\mathbf{v}_M)\right]-\int_{\partial\Omega_s}\mathbf{f}(\mathbf{u})\cdot\mathbf{v}\,\mathrm{d}\partial\Omega_s = 0 \quad \forall\mathbf{v}\in U \quad (25)$$

The results obtained permit making a number of remarks.

Firstly, it can be observed that the failure of real continuous systems is usually due to extreme local values of strain or stress. In the particular system analyzed here, the maximum local strain on the cable is the lowest of all those values corresponding to the same global energy; this permits obtaining the maximum material economy in design and usually leads to system whose collapse under increasing load is due to local body failure while cable failure is, in a certain sense, prevented by the homogeneous strain distribution.

One of the most frequent cases that may occur in applications regards those configurations in which the cable path is rectilinear or piecewise rectilinear, and the particular geometry makes it possible to link the cable to the body either along its whole length or at its end only (e.g. problem with plane symmetry or cable on the surface). It is worthwhile noting that this time $\mathbf{g}(\mathbf{u})$ is constant with respect to η and no differences exist between the conditions describing the solution when the cable is linked to the body, $\mathbf{P}_M = \mathbf{P}_N$, and those describing the solution when the cable is linked at its ends only, $\mathbf{P}_M = \mathbf{P}_O$. However, as will be seen in the next chapter, the two cases show a drastically different behaviour with regard to stability.

The last remark regards the difficulties which can arise in this system in selecting and characterizing the reference configuration. In fact, a configuration in which the body is in its natural state, i.e. the stress field is null everywhere, is usually known and it is convenient to choose this as the reference configuration, even if the cable internal force $t(0)$ corresponding to this configuration is not known. Such a situation occurs because the coupling between cable and body is generally realized with the aim of providing a stress interaction between the two system components and consequently, the configuration chosen as reference is not balanced when no external force is active and the internal force of the cable is known only in the configuration obtained after the coupling, where it assumes the value t^*. On the other hand, in this latter configuration the deformation and stress fields of the body are not known but they can equally be determined by means of the previous condition (25) simply by posing $t(d(\mathbf{u})) = t^*$. Once the displacement body field $\mathbf{u}$ has been evaluated, it becomes possible to obtain the initial force $t(0)$ acting on the cable in the reference configuration by integrating $a(\mathbf{u})$ and by using the constitutive function $\omega(a)$; therefore the complete characterization of the starting configuration is finally obtained.

In numerous interesting engineering applications, the deformations undergone by the body is extremely small, in the sense that $|\nabla\mathbf{u}|$, and consequently its norm in U, is small everywhere, so that it becomes possible to carry out an approximate evaluation of the displacement field dues to t^* by means of a linearized balance condition. In this case the problem assumes the form

$$\int_\Omega [\mathbb{C}(0)((\nabla\mathbf{u})^T+\nabla\mathbf{u})\cdot((\nabla\mathbf{v})^T+\nabla\mathbf{v})/4-\mu_0\mathbf{b}\cdot\mathbf{v}]\,\mathrm{d}\Omega$$

$$= t^*\left[\int_0^m \mathbf{G}\cdot\nabla\mathbf{v}\mathbf{H}_{,\eta}\,\mathrm{d}\eta+\bar{\mathbf{G}}\cdot(\mathbf{v}_N-\mathbf{v}_M)\right]-\int_{\partial\Omega_s}\mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\partial\Omega_s = 0 \quad \forall\mathbf{v}\in U \quad (26)$$

where $\mathbb{C}(\mathbf{D})$ denotes the derivative $\partial\bar{\mathbf{S}}/\partial\mathbf{D}$. It is observed that a linear relation between cable forces t^* and body displacements $\mathbf{u}$ is established when no external forces act.

4. INFINITESIMAL STABILITY

This paragraph states the conditions ensuring infinitesimal stability with respect to a known configuration, characterized by a finite displacement field $\mathbf{u}$ of the body points, by assuming that stability subsists if the second variation of the total potential energy concerning the elastic part of the system, or equivalently in presence of dead loads, is a strictly positive quantity for every possible variation in the displacement field (Tresdell and Noll, 1965). It is noted that this requirement is equivalent to requiring uniqueness for the problem linearized in the neighborhood of the considered configuration (incremental problem).

Taking into account that the second variations of $\mathbf{D}$ and d can be posed in the following form

$$\delta^2\mathbf{D}(\mathbf{u};\mathbf{v},\mathbf{w}) = \tfrac{1}{2}((\nabla\mathbf{v})^T\nabla\mathbf{w}+(\nabla\mathbf{w})^T\nabla\mathbf{v}) \tag{27}$$

$$\delta^2 d(\mathbf{u};\mathbf{v},\mathbf{w}) = \left[\int_0^m \frac{(\mathbf{I}-\mathbf{g}\otimes\mathbf{g})\cdot(\nabla\mathbf{v}\mathbf{H}_{,\eta}\otimes\nabla\mathbf{w}\mathbf{H}_{,\eta})}{|(\mathbf{I}+\nabla\mathbf{u})\mathbf{H}_{,\eta}|}\,\mathrm{d}\eta + \frac{(\mathbf{I}-\bar{\mathbf{g}}\otimes\bar{\mathbf{g}})\cdot((\mathbf{v}_N-\mathbf{v}_M)\otimes(\mathbf{w}_N-\mathbf{w}_M))}{|\mathbf{P}_N-\mathbf{P}_M+\mathbf{u}_N-\mathbf{u}_M|}\right]\Big/ L \tag{28}$$

it is possible to obtain

$$\begin{aligned}\delta^2\Psi(\mathbf{u};\mathbf{v},\mathbf{w}) = {} & +\frac{1}{4}\int_\Omega [\mathbb{C}(\mathbf{D}(\mathbf{u}))((\nabla\mathbf{w})^T+\nabla\mathbf{w}+(\nabla\mathbf{w})^T\nabla\mathbf{u}+(\nabla\mathbf{u})^T\nabla\mathbf{w})\cdot((\nabla\mathbf{v})^T \\ & +\nabla\mathbf{v}+(\nabla\mathbf{v})^T\nabla\mathbf{u})+(\nabla\mathbf{u})^T\nabla\mathbf{v})]\,\mathrm{d}\Omega \quad + c(d(\mathbf{u}))\left[\int_0^m \mathbf{g}(\mathbf{u})\cdot\nabla\mathbf{v}\mathbf{H}_{,\eta}\,\mathrm{d}\eta+\mathbf{g}(\mathbf{u})\cdot(\mathbf{v}_L-\mathbf{v}_M)\right] \\ & \times\left[\int_0^m \mathbf{g}(\mathbf{u})\cdot\nabla\mathbf{w}\mathbf{H}_{,\eta}\,\mathrm{d}\eta+\mathbf{g}(\mathbf{u})\cdot(\mathbf{w}_N-\mathbf{w}_M)\right] \quad +\frac{1}{2}\int_\Omega \bar{\mathbf{S}}(\mathbf{D}(\mathbf{u}))\cdot((\nabla\mathbf{v})^T\nabla\mathbf{w}+(\nabla\mathbf{w})^T\nabla\mathbf{v})\,\mathrm{d}\Omega \\ & + t(d(\mathbf{u}))\left[\int_0^m \frac{(\mathbf{I}-\mathbf{g}\otimes\mathbf{g})}{|(\mathbf{I}+\nabla\mathbf{u})\mathbf{H}_{,\eta}|}\cdot(\nabla\mathbf{v}\mathbf{H}_{,\eta}\otimes\nabla\mathbf{w}\mathbf{H}_{,\eta})\,\mathrm{d}\eta \right. \\ & \left. + \frac{(\mathbf{I}-\bar{\mathbf{g}}\otimes\bar{\mathbf{g}})\cdot((\mathbf{v}_N-\mathbf{v}_M)\otimes(\mathbf{w}_N-\mathbf{w}_M))}{|\mathbf{P}_N-\mathbf{P}_M+\mathbf{u}_N-\mathbf{u}_M|}\right] = 0 \quad \forall(\mathbf{v},\mathbf{w})\in U\end{aligned} \tag{29}$$

where $\mathbb{C}(\mathbf{D})$ denotes, as previously, the derivative $\partial\bar{\mathbf{S}}/\partial\mathbf{D}$ while $c(d)$ denotes the derivative $\mathrm{d}t/\mathrm{d}d$. In conclusion, the second variation defines a bilinear form $q(\mathbf{u}): U\times U\to\mathbb{R}$, and can be rewritten in the compact form as

$$\delta^2\Psi(\mathbf{u};\mathbf{v},\mathbf{w}) = q(\mathbf{u})(\mathbf{v},\mathbf{w}) \tag{30}$$

consequently, the solution described by the displacement field u is stable, and the solution of the problem linearized in the neighborhood of $\mathbf{u}$ exists and is unique, if $q(\mathbf{u})$ is coercive, i.e. if a real value $k(\mathbf{u}) > 0$ such that

$$k(\mathbf{u}) = \inf\frac{q(\mathbf{u})(\mathbf{v},\mathbf{v})}{\|\mathbf{v}\|_U^2} > 0 \quad \forall\,\mathbf{v}\neq 0 \tag{31}$$

exists.

It can be observed that the bilinear form $q(\mathbf{u})$ consists in the sum of four terms (see 29). The first two terms are usually positive definite on U because, for stable material, the constitutive quantities $\mathbb{C}$ and c are two positive definite and bounded operators for the examined displacement field $\mathbf{u}$ satisfying the boundary kinematical conditions. The third term, depending on the stress field provided by the cable and the external forces of the body, may be not positive definite and, in this case, may represent the cause of instability. Finally, the last term, proportional to the generally positive cable internal force t, is only positive semi-definite on U and shows its stabilizing effect only for those displacement fields $\mathbf{v}$ possessing components which are orthogonal to the cable path, as implied by the projector $(\mathbf{I}-\mathbf{g}\otimes\mathbf{g})$.

Attention was previously focused on the case of configurations with a rectilinear trajectory of the cable, showing that the equilibrium conditions are not affected by the type of connection existing between cable and body. It will now be showed that the type of connection plays a determinant role with regard to stability by comparing two different limit situations, one furnished by a cable path completely linked to the body ($\mathbf{P}_M = \mathbf{P}_N$) and the other furnished by a cable linked to the body at its ends only ($\mathbf{P}_M = \mathbf{P}_O$). In particular, taking into account that the unit vector $\mathbf{g}$ of the former situation is constant and coincides with the unit vector $\bar{\mathbf{g}}$ of the second situation, so that $\nabla\mathbf{v}\bar{\mathbf{g}}$ is a square-integrable function with respect to η, it is possible to apply Holder's inequality (Brezis, 1983) in the following form:

$$|(\mathbf{I}-\bar{\mathbf{g}}\otimes\bar{\mathbf{g}})(\mathbf{v}_N-\mathbf{v}_O)|^2 = \left|\int_0^n (\mathbf{I}-\bar{\mathbf{g}}\otimes\bar{\mathbf{g}})\nabla\mathbf{v}\mathbf{H}_{,\eta}\,\mathrm{d}\eta\right|^2$$

$$\leqslant \int_0^n |(\mathbf{I}+\nabla\mathbf{u})\mathbf{H}_{,\eta}|\,\mathrm{d}\eta \int_0^n \frac{|(\mathbf{I}-\bar{\mathbf{g}}\otimes\bar{\mathbf{g}})\nabla\mathbf{v}\mathbf{H}_{,\eta}|^2}{|(\mathbf{I}+\nabla\mathbf{u})\mathbf{H}_{,\eta}|}\mathrm{d}\eta \quad (32)$$

and therefore, by evaluating the first integral, the sought inequality is achieved

$$\frac{|(\mathbf{I}-\bar{\mathbf{g}}\otimes\bar{\mathbf{g}})(\mathbf{v}_N-\mathbf{v}_0)|^2}{|\mathbf{P}_N-\mathbf{P}_O+\mathbf{u}_N-\mathbf{u}_O|} \leqslant \left[\int_0^n \frac{|(\mathbf{I}-\mathbf{g}\otimes\mathbf{g})\nabla\mathbf{v}\mathbf{H}_{,\eta}|^2}{|(\mathbf{I}+\nabla\mathbf{u})\mathbf{H}_{,\eta}|}\mathrm{d}\eta\right]^2 \quad (33)$$

from which it can be concluded that $k(\mathbf{u})$ evaluated in the former situation, for which $\mathbf{P}_M = \mathbf{P}_N$, is greater or equal to $k(\mathbf{u})$ evaluated in the second situation, where $\mathbf{P}_M = \mathbf{P}_O$, and consequently, even if the same stress state is active, it is possible to reduce the risk of instability by increasing the connection degree between cable and body.

As already mentioned, many real problems can be analyzed with a completely satisfactory approximation, by means of a linear theory which establishes a proportional relation between the body stress field and its causes, i.e. cable force and external actions (see eqn (26)). In order to analyze the effects of the cable force only on the equilibrium stability, it is assumed that the examined configuration coincides with the reference configuration, the external forces are absent and the stress field on the body, this time described by the Cauchy stress $\mathbf{T}(X_i)$, is proportional to the force t acting on the cable, i.e. $\mathbf{T} = t\mathbf{T}_t(X_i)$. In this case, the second variation of the energy can be decomposed as follows:

$$q(\mathbf{v},\mathbf{w}) = q_0(\mathbf{v},\mathbf{w}) + tq_t(\mathbf{v},\mathbf{w}) \quad (34)$$

where the dependence on $\mathbf{u} = 0$ has been omitted and the two terms have the following expressions:

$$q_0(\mathbf{v},\mathbf{w}) = \int_\Omega \tfrac{1}{4}[\mathbb{C}((\nabla\mathbf{v})^T+\nabla\mathbf{v})\cdot((\nabla\mathbf{w})^T+\nabla\mathbf{w})]\,\mathrm{d}\Omega$$

$$+\frac{c}{L}\left[\int_0^m \mathbf{G}\cdot\nabla\mathbf{v}\mathbf{H}_{,\eta}\,\mathrm{d}\eta+\bar{\mathbf{G}}\cdot(\mathbf{v}_N-\mathbf{v}_M)\right]\left[\int_0^m \mathbf{G}\cdot\nabla\mathbf{w}\mathbf{H}_{,\eta}\,\mathrm{d}\eta+\bar{\mathbf{G}}\cdot(\mathbf{w}_N-\mathbf{w}_M)\right] \quad (35)$$

$$q_t(\mathbf{v},\mathbf{w}) = \int_\Omega \tfrac{1}{2}[\mathbf{T}_t\cdot((\nabla\mathbf{v})^T\nabla\mathbf{w}+(\nabla\mathbf{w})^T\nabla\mathbf{v})]\,\mathrm{d}\Omega$$
$$+\left[\int_0^m \frac{(\mathbf{I}-\mathbf{G}\otimes\mathbf{G})(\nabla\mathbf{v}\mathbf{H}_{,\eta}\otimes\nabla\mathbf{w}\mathbf{H}_{,\eta})}{|\mathbf{H}_{,\eta}|}\,\mathrm{d}\eta+\frac{(\mathbf{I}-\bar{\mathbf{G}}\otimes\bar{\mathbf{G}})((\mathbf{v}_N-\mathbf{v}_M)\otimes(\mathbf{w}_N-\mathbf{w}_M))}{|\mathbf{P}_N-\mathbf{P}_M|}\right] \quad (36)$$

The first term depends on the stiffness of the materials in the neighborhood of the assigned configuration and, if $\mathbb{C}$ and c are positive definite and bounded, $k_0 > 0$ such that

$$k_0 = \inf\frac{q_0(\mathbf{v},\mathbf{v})}{\|\mathbf{v}\|^2_U} > 0 \quad \forall\mathbf{v}\neq 0 \quad (37)$$

exists, in the usual case $t > 0$, the sole instability cause may therefore be located in the term q_t related to the existence of a stress field different from zero in the examined configuration. In particular, q_t is exclusively related to the body geometry, which determines the stress field due to a unit force on the cable, and to the geometry of the cable path, but it does not depend on the intensity t of the cable force. So that if the system geometry makes q_t positive definite, the system will be stable for every value of t, at least in the linear range between $\mathbf{T}$ and t, while if the system geometry makes q_t non-definite, a particular value t_{cr} providing instability will surely exist. By denoting with

$$k_t = \inf\frac{q_t(\mathbf{v},\mathbf{v})}{\|\mathbf{v}\|^2_U} > 0 \quad \forall\mathbf{v}\neq 0 \quad (38)$$

and considering that

$$q(\mathbf{v},\mathbf{v}) \geqslant \|\mathbf{v}\|^2_U(k_0+tk_t) \quad (39)$$

it is possible to conclude that stability is ensured when $k_t > 0$ or when $k_t < 0$ and $t < -k_0/k_t$. However, the global stiffness of the system may be softened with respect to the stiffness evaluated for the natural configuration and instability phenomena imputable to external forces may occur prematurely when $k_t < 0$, because k_0 becomes an upper bound for the coercivity constant k in the bilinear form q, even if instability does not occur for the considered force t.

In real cases, the purposes of this investigation consist in searching for that cable path which minimizes the unstabilizing effects between all those paths satisfying the design requirements. The problem can be often simplified by carrying out the analysis on subspaces of U obtained by eliminating those deformations which are not of interest because they are related to critical values of t that are not compatible with the initial assumption of linearity for the fundamental state (as in the example reported below).

5. APPLICATION

The example examines a system consisting of a rectilinear beam and a cable arranged parallelly to the beam axis in order to analyze the consequences of cable stretching on stability, in the range of small deformations. This technique is often used in structural application to reduce the maximum stress value caused by external actions on steel or concrete girders. The considered case involves a simple geometry but it equally permits demonstrating some qualitative aspects of the problem.

A prismatic solid occupying the region $\Omega = \{X_\alpha\mathbf{A}_\alpha+\zeta\mathbf{A}_3;\ \alpha = 1, 2;\ (X_\alpha)\in S\subset\mathbb{R}^2;\ \zeta\in[0, L]\}$ in the reference configuration is considered and its cross section S is a doubly symmetric domain, with respect to the axis X_α. A curve $\mathscr{H} = \{X_1 = 0,\ X_2 = s,\ X_3 = \eta;\ \eta\in[0, L]\}$ is defined on the body and traces the following path in the reference configuration

$$\mathbf{H}(\eta) = H_i(\eta)\mathbf{A}_i = s\mathbf{A}_2 + \eta\mathbf{A}_3 \quad \eta \in [0, L] \tag{40}$$

and its derivative is furnished by $\mathbf{H}_{,\eta}(\eta) = \mathbf{G}(\eta) = \mathbf{A}_3$. In the reference of configuration an internal force t acts on the cable and a stress field $\mathbf{T} = -t(1/A + sX_2/J_2)(\mathbf{A}_3 \otimes \mathbf{A}_3)$ on the body, having denoted by A the area of the surface S and by J_2 its moment of inertia evaluated with respect to the axis X_1. It is also assumed that a rigid apparatus able to provide equilibrium is posed at the beam ends or, equivalently, that the local phenomena occurring near the anchorages may be neglected. The described state of the system has been achieved in the range of validity of a linear theory in the neighborhood of the natural state of the beam.

The described stress field can be generated either by a cable anchored at the end points $\mathbf{P}_O = s\mathbf{A}_2$, $\mathbf{P}_N = s\mathbf{A}_2 + L\mathbf{A}_3$ and constrained to follow the path $\mathbf{H}$ ($\mathbf{P}_M = \mathbf{P}_N$) or by a cable anchored at the end points only and free at the other points ($\mathbf{P}_M = \mathbf{P}_O$). Initially the case of a slipping cable constrained on the curve $\mathbf{H}$ is considered.

The infinitesimal stability of the assigned configuration is analyzed by describing the behaviour of the beam by means of the following approximate displacement field (Vlasov, 1961):

$$\mathbf{u}(X_\alpha; \zeta) = \mathbf{u}_0(\zeta) + [\phi_{0,3}(\zeta)\psi(X_\alpha) - \mathbf{Y}\cdot\mathbf{u}_{0,3}(\zeta)]\mathbf{A}_3 + \phi_0(\zeta)\mathbf{A}_3 \times \mathbf{Y} \tag{41}$$

where $\mathbf{u}_0 = u_{0\alpha}(\zeta)\mathbf{A}_\alpha$ denotes the transversal displacements of the axis line $(0, 0, \zeta)$, $\mathbf{Y} = \mathbf{X}_\alpha\mathbf{A}_\alpha$ is the projection of the position vector on the section S. $\phi_0(\zeta)$ denotes the angle of rigid rotation around $\mathbf{A}_3$ and $\psi(X_\alpha)$ is the De Saint Venant warping function. The expressions of the displacement gradient is the following ($\nabla\psi = \psi_{,\alpha}\mathbf{A}_\alpha$):

$$\nabla\mathbf{u} = \phi_0(\mathbf{A}_2 \otimes \mathbf{A}_1 - \mathbf{A}_1 \otimes \mathbf{A}_2) + \mathbf{A}_3 \otimes (\phi_{0,3}\nabla\psi - \mathbf{u}_{0,3}) + (\mathbf{u}_{0,3} + \phi_{0,3}\mathbf{A}_3 \times \mathbf{Y}) \otimes \mathbf{A}_3 + [\phi_{0,33}\psi - \mathbf{Y}\cdot\mathbf{u}_{0,33}](\mathbf{A}_3 \otimes \mathbf{A}_3) \tag{42}$$

Such a displacement field maps $\mathbf{H}$ into the curve

$$\mathbf{h}(\eta) = \mathbf{u}_0(\eta) + s[\mathbf{A}_2 - \phi_{0,3}(\eta)\mathbf{A}_1] + [\phi_{0,3}(\eta)\psi(0, s) - su_{02,3}(\eta)]\mathbf{A}_3 \tag{43}$$

while the term $\nabla\mathbf{u}\mathbf{H}_{,\eta}$ assumes the form

$$\nabla\mathbf{u}(0, s, \eta)\mathbf{H}_{,\eta}(\eta) = \mathbf{u}_{0,3}(\eta) - s\phi_{0,3}(\eta)\mathbf{A}_1 + [\phi_{0,33}(\eta)\psi(0, s) - su_{02,33}(\eta)]\mathbf{A}_3 \tag{44}$$

The material forming the beam is isotropic and homogeneous; E is its normal elastic modulus and G its tangential elastic modulus. Furthermore, J_1 denotes the moment of inertia with respect to the axis X_2, β_α the inertial radii $\sqrt{J_\alpha/A}$, β_p the polar radius $\sqrt{(J_1 + J_2)/A}$, J, J_ψ, $e\bar{J}$ are the following quantities:

$$J = \int_S (\psi_{,1} - X_2)^2 + (\psi_{,2} + X_1)^2 \, \mathrm{d}S \tag{45}$$

$$J_\psi = \int_S \psi^2 \, \mathrm{d}S \tag{46}$$

$$\bar{J} = \int_S X_1 X_2 \psi \, \mathrm{d}S / J_2 \tag{47}$$

and $\beta_\psi = \sqrt{J_\psi/A}$. The considered displacement model provides the following expression for the bilinear form (34):

$$q(\mathbf{u},\mathbf{u}) = [EJ_\alpha - t(\beta_\alpha)^2]\int_0^L u_{0\alpha,33}^2\,\mathrm{d}\zeta + [EJ_\psi - t\beta_\psi^2]\int_0^L \phi_{0,33}^2\,\mathrm{d}\zeta$$
$$+[GJ - t\beta_p^2 + ts^2]\int_0^L \phi_{0,3}^2\,\mathrm{d}\zeta + 2ts\bar{J}\int_0^L \phi_{0,33}u_{01,33}\,\mathrm{d}\zeta$$
$$+\frac{c}{L}[\psi(0,s)(\phi_{0,3}(L)-\phi_{0,3}(0)) - s(u_{02,3}(L)-u_{02,3}(0))]^2 \quad (48)$$

When the boundary conditions prevent the transversal displacements and the rigid rotations around the axis, the solution can easily be obtained. The constraint introduced by the reduced displacement field led to a unidimensional problem and the regularity previously required can be obtained by assuming that the components of $\mathbf{u}_0$ and ϕ_0 satisfy the boundary conditions and have square integrable second derivatives, i.e. $(\mathbf{u}_0, \phi_0) \in H_0^2(0, L)$; they consequently can be expressed by means of the complete orthonormal series $\{\xi_i(\zeta) = \sqrt{2/L}\sin(i\pi\zeta/L); i \in \mathbb{N}/0\}$. By denoting with $\bar{u}_{0\alpha i}$ the scalar products between $u_{0\alpha}(\zeta)$ and $\xi_i(\zeta)$ and with $\bar{\phi}_{0i}$ the scalar products between $\phi_0(\xi)$ and $\xi_i(\zeta)$, the following expression of the bilinear form is obtained:

$$q(\mathbf{u},\mathbf{u}) = [EJ_\alpha - t\beta_\alpha^2]\frac{i^4\pi^4}{L^4}[\bar{u}_{0\alpha i}]^2 + [EJ_\psi - t\beta_\phi^2]\frac{j^4\pi^4}{L^4}[\bar{\phi}_{0j}]^2$$
$$+[GJ - t\beta_p^2 + ts^2]\frac{k^2\pi^2}{L^2}[\bar{\phi}_{0k}]^2 + 2ts\bar{J}\frac{i^4\pi^4}{L^4}[\bar{\phi}_{0i}][\bar{u}_{01i}]$$
$$+\frac{8c\pi^2}{L^4}\left[\frac{1-(-1)^i}{2}i\psi(0,s)\bar{\phi}_{0i} - \frac{1-(-1)^j}{2}js\bar{u}_{02j}\right]^2 \quad i,j,k \in \{\mathbb{N}/0\} \quad (49)$$

From (49) it is possible to deduce the critical values of the cable force, for which the form q is no longer positive definite. It may be observed that, if the length L is great enough, the more significant term for determining the minimum value of t which makes the form q non positive definite is the diagonal term $[GJ - t\beta_p^2 + ts^2]$ related to the torsional deformation described by ϕ_0 while the bending like deformations are related to extremely high critical values of t which are not compatible with the initial assumption of small deformations for the fundamental state. Therefore, the investigation can be limited to that subspace of U consisting of only those deformations corresponding to dangerous values of t; in this case, the position of the curve $\mathbf{H}$, described by the sole parameter s, proves to be a determining factor for stability and it becomes possible to define a set of cable positions for which instability will never occur. Going back to the previously mentioned more significative term, it can be noted that this term becomes null when $t = GJ/(\beta_p^2 - s^2)$, it follows that torsional instability phenomena can occur when the cable is at a distance of less than β_p from the centroid, while they are prevented in the other cases.

By applying the same deformation model to a beam whose cable is constrained at its ends only, the following different expression of the bilinear form q is achieved:

$$q(\mathbf{u},\mathbf{u}) = [EJ_\alpha - t\beta_\alpha^2]\int_0^L u_{0\alpha,33}^2\,\mathrm{d}\zeta + [EJ_\psi - t\beta_\psi^2]\int_0^L \phi_{0,33}^2\,\mathrm{d}\zeta$$
$$-t\int_0^L u_{0\alpha,3}u_{0\alpha,3}\,\mathrm{d}\zeta + [GJ - t\beta_p^2]\int_0^L \phi_{0,3}^2\,\mathrm{d}\zeta - 2ts\int_0^L \phi_{0,3}u_{01,3}\,\mathrm{d}\zeta$$
$$-2ts\bar{J}\int_0^L \phi_{0,33}u_{01,33}\,\mathrm{d}\zeta + t[(u_{01}(L)-u_{01}(0)-s\phi_0(L)+s\phi_o(0))^2 + (u_{02}(L)-u_{02}(0))^2]/L$$
$$+\frac{c}{L}[\psi(0,s)(\phi_{0,3}(L)-\phi_{0,3}(0)) - s(u_{02,3}(L)-u_{02,3}(0))]^2 \quad (50)$$

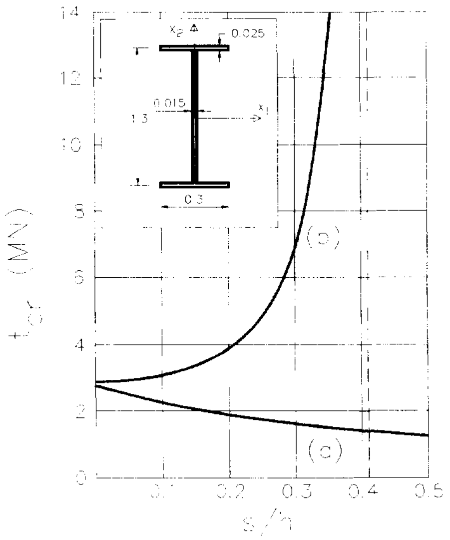

Fig. 2. Critical traction force on the cable.

from which it is possible to deduce the critical values for the cable force by means of a procedure analogous to the previous. In this latter case, in addition to a reduction in the minimum critical values, a remarkable coupling between the terms containing ϕ_0 and the terms containing u_{01} occurs; this provides critical force t connected to bending-torsional deformation and the structural behaviour of this system is also qualitatively different from that observed in the former case.

A numerical application is reported. This involves a steel beam ($E = 2.1 \cdot 10^5$ MPa) with a geometry usually employed in composite beams, for which the prestressing is introduced before the cast of the upper concrete slab. In Fig. 2 the critical values of t obtained by varying the cable position s are reported for the beam with the section depicted in the figure (lengths in metres), with a ratio section height h vs length equal to 10, $G/E = 0.4$ and $c/EA = 0.1$. In the case of cable constrained at the ends only (curve (a)) extremely low values of critical force t, mainly related to bending deformation, are obtained while a remarkably better situation is provided in the other case of cable connected along the whole beam (curve (b)). The dashed line denotes the vertical asymptote of the curve (b), that can be obtained with acceptable approximation from the relation $s = \beta_p$, if the length L is notably longer than the cross section dimensions.

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